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CEVA'S AND MENELAUS' THEOREMS CHARACTERIZE THE HYPERBOLIC GEOMETRY AMONG HILBERT GEOMETRIES

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ABSTRACT. If a Hilbert geometry satisfies a rather weak version of either Ceva's or Menelaus' theorem for every triangle, then it is hyperbolic.

1. Introduction

Hilbert geometries, as introduced by David Hilbert in 1899 [6], are natural generalizations of hyperbolic geometry. It raised the question immediately whether some Hilbert geometries are isomorphic to hyperbolic geometry, and several such results were born over the years. For a recent survey of this subject, see [5].

In this short note we prove that rather weak hyperbolic versions of the classical results known as Theorem of Menelaus, resp. Ceva, provide us with a criterion which makes Hilbert geometry hyperbolic.

2. Preliminaries

Points of $\mathbb{R}^n$ are denoted as $\mathbf{a}, \mathbf{b}, \dots$; the line through different points $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{ab}$, the open segment with endpoints $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\overline{\mathbf{ab}}$. Non-degenerate triangles are called *trigons*.

For given different points $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^n$, and points $\mathbf{x}, \mathbf{y} \in \mathbf{pq}$ one has the unique linear combinations $\mathbf{x} = \lambda_1 \mathbf{p} + \mu_1 \mathbf{q}$, $\mathbf{y} = \lambda_2 \mathbf{p} + \mu_2 \mathbf{q}$ $(\lambda_1 + \mu_1 = 1,\ \lambda_2 + \mu_2 = 1)$ which allows to define the *cross ratio* $(\mathbf{p}, \mathbf{q}; \mathbf{x}, \mathbf{y}) = \frac{\mu_1 \lambda_2}{\lambda_1 \mu_2}$ of the points $\mathbf{p}, \mathbf{q}, \mathbf{x}$ and $\mathbf{y}$ [1, page 243] provided that $\lambda_1 \mu_2 \neq 0$.

Let $\mathcal{H}$ be an open, strictly convex set in $\mathbb{R}^n$ $(n \geq 2)$ with boundary $\partial\mathcal{H}$. The function $d_{\mathcal{H}} \colon \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ defined by

$$(1) \qquad d_{\mathcal{H}}(\mathbf{x}, \mathbf{y}) = \begin{cases} 0, & \text{if } \mathbf{x} = \mathbf{y}, \\ \frac{1}{2}\big|\ln|(\mathbf{p}, \mathbf{q}; \mathbf{x}, \mathbf{y})|\big|, & \text{if } \mathbf{x} \neq \mathbf{y}, \text{ where } \overline{\mathbf{pq}} = \mathcal{H} \cap \mathbf{xy}, \end{cases}$$

is a metric on $\mathcal{H}$ [1, page 297], and is called the *Hilbert metric on* $\mathcal{H}$. The pair $(\mathcal{H}, d_{\mathcal{H}})$ is called the *Hilbert geometry* given in $\mathcal{H}$.

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Note that two Hilbert geometries are isomorphic if there is a projective map between their sets of points, because all the defining conditions of a Hilbert geometry is projective invariant.

The generalized Cayley–Klein model [7] of the hyperbolic geometry $\mathbb{H}^n$ is, in fact, a Hilbert geometry $(\mathcal{E}, d_{\mathcal{E}})$ of special kind given by an ellipsoid $\mathcal{E}$. By the above note, all these are isomorphic to each other and to $\mathbb{H}^n$. It is easy to see that a Hilbert geometry is hyperbolic if and only if $\mathcal{H}$ is an ellipsoid.

The following results become useful in the next section. The first one is a week version of [3, Lemma 12.1, pp. 226].

Lemma 2.1. *A bounded open convex set $\mathcal{H}$ in $\mathbb{R}^n$ $(n \geq 2)$ is an ellipsoid if and only if every section of it by any 2-dimensional plane is an ellipse.*

Lemma 2.2. *For any nonempty open convex set $\mathcal{H}$ in the plane, which is not an ellipse, there is an ellipse $\mathcal{E}$ such that $\partial\mathcal{E} \cap \partial\mathcal{H}$ has at least six different points, and $\mathcal{E} \setminus \mathcal{H}$ is nonempty.*

Proof. By [2, Theorem 2] of F. John, there is an ellipse $\mathcal{E}$ containing $\mathcal{H}$ and having at least 3 contact points, i.e., $|\partial\mathcal{H} \cap \partial\mathcal{E}| \geq 3$.

As $\mathcal{H}$ is not an ellipse, $\mathcal{E} \setminus \mathcal{H}$ is not empty, but contains an open set of points.

If $|\partial\mathcal{H} \cap \partial\mathcal{E}| \geq 6$, then the statement is proved.

If $|\partial\mathcal{H} \cap \partial\mathcal{E}| \leq 5$, then shrinking the ellipse $\mathcal{E}$ with a homothety χ at a center in $\mathcal{H}$ with coefficient $1-\varepsilon$, where $\varepsilon > 0$ is small enough, moves the points in $\partial\mathcal{H} \cap \partial\mathcal{E}$ into $\mathcal{H}$ and the resulting ellipse $\chi(\mathcal{E})$ proves the lemma. □

Let $\mathbf{a}, \mathbf{b}$ be different points in $\mathcal{H}$ and let $\mathbf{c}$ be in $(\mathbf{ab} \cap \mathcal{H}) \setminus \{\mathbf{b}\}$. The so called *hyperbolic ratio*[1] $\langle \mathbf{a}, \mathbf{b}; \mathbf{c}\rangle_{\mathcal{H}}$ of the triple $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is defined by

$$\langle \mathbf{a}, \mathbf{b}; \mathbf{c}\rangle_{\mathcal{H}} = \begin{cases} -\frac{\sinh d_{\mathcal{H}}(\mathbf{c},\mathbf{a})}{\sinh d_{\mathcal{H}}(\mathbf{b},\mathbf{c})}, & \text{if } \mathbf{c} \in \overline{\mathbf{ab}}, \\ \frac{\sinh d_{\mathcal{H}}(\mathbf{c},\mathbf{a})}{\sinh d_{\mathcal{H}}(\mathbf{b},\mathbf{c})}, & \text{otherwise.} \end{cases} \tag{2}$$

Lemma 2.3. *Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be collinear points in a Hilbert geometry $\mathcal{H}$, and let $\{\mathbf{p}, \mathbf{q}\} = \mathbf{ab} \cap \partial\mathcal{H}$, such that $\mathbf{a}$ separates $\mathbf{p}$ and $\mathbf{b}$. Set an Euclidean coordinate system on $\mathbf{ab}$ such that the coordinates of $\mathbf{p}$ and $\mathbf{a}$ are 0 and 1, respectively. Let q, b and c, with assumptions $q > b > 1$ and $0 < c < q$, be the coordinates of $\mathbf{q}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. Then we have*

$$|\langle \mathbf{a}, \mathbf{b}; \mathbf{c}\rangle_{\mathcal{H}}| = \frac{|c-b|}{|c-1|\sqrt{b}} \sqrt{1 + \frac{b-1}{q-b}}. \tag{3}$$

[1] The name 'hyperbolic ratio' comes from the hyperbolic sine function in the definition.

Proof. Using (2) we can write

$$|\langle \mathbf{a},\mathbf{b},\mathbf{c}\rangle_{\mathcal{H}}| = \frac{\sinh\left(\frac{\operatorname{sign}(c-b)}{2}\ln\frac{c(q-b)}{b(q-c)}\right)}{\sinh\left(\frac{\operatorname{sign}(1-c)}{2}\ln\frac{q-c}{c(q-1)}\right)} = \frac{\left(\frac{c(q-b)}{b(q-c)}\right)^{\frac{\operatorname{sign}(c-b)}{2}} - \left(\frac{b(q-c)}{c(q-b)}\right)^{\frac{\operatorname{sign}(c-b)}{2}}}{\left(\frac{q-c}{c(q-1)}\right)^{\frac{\operatorname{sign}(1-c)}{2}} - \left(\frac{c(q-1)}{q-c}\right)^{\frac{\operatorname{sign}(1-c)}{2}}}$$
$$= \frac{\left((c(q-b))^{\operatorname{sign}(c-b)} - (b(q-c))^{\operatorname{sign}(c-b)}\right)(c(q-1)(q-c))^{\frac{\operatorname{sign}(1-c)}{2}}}{\left((q-c)^{\operatorname{sign}(1-c)} - (c(q-1))^{\operatorname{sign}(1-c)}\right)(b(q-c)c(q-b))^{\frac{\operatorname{sign}(c-b)}{2}}}.$$

Now we have three cases to handle: $0<c<1$, $1<c<b$ and $b<c<q$.

If $0<c<1$, hence $\operatorname{sign}(1-c)=1$ and $\operatorname{sign}(c-b)=-1$, then

$$|\langle \mathbf{a},\mathbf{b},\mathbf{c}\rangle_{\mathcal{H}}| = \frac{\left((c(q-b))^{-1} - (b(q-c))^{-1}\right)(c(q-1)(q-c))^{\frac{1}{2}}}{((q-c)-c(q-1))\,(b(q-c)c(q-b))^{\frac{-1}{2}}}$$
$$= \frac{(b-c)q\,(q-1)^{\frac{1}{2}}}{q(1-c)\,(b(q-b))^{\frac{1}{2}}} = \frac{(b-c)}{(1-c)\sqrt{b}}\sqrt{\frac{q-1}{q-b}}.$$

The other two cases can be calculated in the same way, hence this lemma is proved. □

3. Ceva's and Menelaus' Theorems in Hilbert geometry

Ceva's and Menelaus' Theorems are valid also in the hyperbolic geometry [7, pp. 467-468] if the hyperbolic ratio (2) is used. For the sake of completeness we recall here the necessary terms and the Theorems themselves.

Let $\mathbf{a},\mathbf{b},\mathbf{c}$ be the vertices of a trigon. By a *triplet* $(\mathbf{c}',\mathbf{a}',\mathbf{b}')$ *of the trigon* $\mathbf{abc}$ we mean three points $\mathbf{c}',\mathbf{a}'$ and $\mathbf{b}'$ being respectively on the straight lines $\mathbf{ab}$, $\mathbf{bc}$ and $\mathbf{ca}$. A triplet $(\mathbf{c}',\mathbf{a}',\mathbf{b}')$ of the trigon $\mathbf{abc}$ is called

(p1) a *Menelaus triplet* if the points $\mathbf{c}'$, $\mathbf{a}'$ and $\mathbf{b}'$ are collinear, and
(p2) a *Ceva triplet* if the lines $\mathbf{aa}'$, $\mathbf{bb}'$ and $\mathbf{cc}'$ are concurrent.

A triple (α,β,γ) of real numbers is

(n1) of *Menelaus type* if $\alpha\cdot\beta\cdot\gamma=-1$, and
(n2) of *Ceva type* if $\alpha\cdot\beta\cdot\gamma=+1$.

With these terms, *Hyperbolic Ceva's Theorem* states that a triplet $(\mathbf{c}',\mathbf{a}',\mathbf{b}')$ of a trigon $\mathbf{abc}$ in $\mathbb{H}^n$ is of Ceva type if and only if the triple $(\langle\mathbf{a},\mathbf{b},\mathbf{a}'\rangle_{\mathbb{H}^n}, \langle\mathbf{b},\mathbf{c},\mathbf{a}'\rangle_{\mathbb{H}^n}, \langle\mathbf{c},\mathbf{a},\mathbf{b}'\rangle_{\mathbb{H}^n})$ is of Ceva type. The *Hyperbolic Menelaus' Theorem* asserts that a triplet $(\mathbf{c}',\mathbf{a}',\mathbf{b}')$ of a trigon $\mathbf{abc}$ in $\mathbb{H}^n$ is of Menelaus type if and only if the triple $(\langle\mathbf{a},\mathbf{b},\mathbf{c}'\rangle_{\mathbb{H}^n}, \langle\mathbf{b},\mathbf{c},\mathbf{a}'\rangle_{\mathbb{H}^n}, \langle\mathbf{c},\mathbf{a},\mathbf{b}'\rangle_{\mathbb{H}^n})$ is of Menelaus type.

In Hilbert geometries neither of these theorems are valid, in general. Instead, quite weak versions of them characterize the hyperbolic the Hilbert geometry.

The next theorem readily generalizes R. Guo's result on medians of a trigon [4]. The way of proving[2] goes basically along the idea of R. Guo.

Theorem 3.1 (Ceva type characterization)**.** *If for every trigon* $\mathbf{abc}$ *in* $\mathcal{H}$ *there is a Ceva triplet* $(\mathbf{c}', \mathbf{a}', \mathbf{b}')$ *such that the triple* $(\langle \mathbf{a}, \mathbf{b}, \mathbf{c}'\rangle_{\mathcal{H}}, \langle \mathbf{b}, \mathbf{c}, \mathbf{a}'\rangle_{\mathcal{H}}, \langle \mathbf{c}, \mathbf{a}, \mathbf{b}'\rangle_{\mathcal{H}})$ *is of Ceva type, then the Hilbert geometry* $(\mathcal{H}, d_{\mathcal{H}})$ *is the hyperbolic geometry.*

Proof. We have to show that $\mathcal{H}$ is an ellipsoid. By Lemma 2.1 we only need to work in the plane, therefore $\mathcal{H}$ is in a plane from now on in this proof.

Assume that $\partial\mathcal{H}$ is not an ellipse. Then, by Lemma 2.2, we have an ellipse $\mathcal{E}$ such that $\partial\mathcal{E} \cap \partial\mathcal{H}$ has at least six different points $\mathbf{p}_i$ $(i = 1, \dots, 6)$, and $\mathcal{E} \setminus \mathcal{H}$ contains some some open sets.

Choose a point $\mathbf{p}_0 \in \mathcal{E} \setminus \mathcal{H}$ such that a neighborhood $\mathcal{U}$ of it is in $\mathcal{E} \setminus \mathcal{H}$.

The lines $\mathbf{p}_0\mathbf{p}_i$ $(i = 1, 2, 3, 4, 5)$ are clearly pairwise different, therefore exactly one of them separates the four remaining points, so that exactly two of those points are on both sides of it. Assume that the indexes were chosen in such a way that this separating line is $\mathbf{p}_0\mathbf{p}_3$, points $\mathbf{p}_1$ and $\mathbf{p}_2$ are on its left side, $\mathbf{p}_4$ and $\mathbf{p}_5$ are on its right side, and the segment $\overline{\mathbf{p}_1\mathbf{p}_4}$ meets $\overline{\mathbf{p}_2\mathbf{p}_5}$ in a point $\mathbf{a} \in \mathcal{E} \cap \mathcal{H}$. If $\mathbf{a} \in \overline{\mathbf{p}_0\mathbf{p}_3}$, then move the point $\mathbf{p}_0$ a little bit over so that it remains in $\mathcal{U}$ and $\mathbf{p}_0\mathbf{p}_3$ separates $\overline{\mathbf{p}_1\mathbf{p}_2}$ and $\overline{\mathbf{p}_4\mathbf{p}_5}$. In this way we have $\mathbf{a} \notin \overline{\mathbf{p}_0\mathbf{p}_3}$.

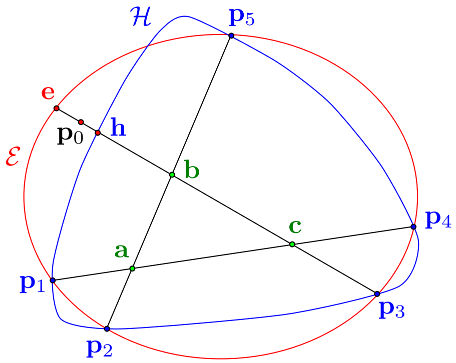

FIGURE 1. The triangle given by the contact points.

Let $\mathbf{b} = \overline{\mathbf{p}_0\mathbf{p}_3} \cap \overline{\mathbf{p}_1\mathbf{p}_4}$, and $\mathbf{c} = \overline{\mathbf{p}_0\mathbf{p}_3} \cap \overline{\mathbf{p}_2\mathbf{p}_5}$. Then $\mathbf{abc}$ is a trigon.

[2]We point out that in Guos's proof the procedure is inadequate to find proper ellipse with six appropriate points on the border of the Hilbert geometry and it does not allow considering general Ceva triplets, because some points of some Ceva triplets might fall outside of the ellipse.

Observe that the open segments $\overline{\mathbf{p}_1\mathbf{p}_4}$ and $\overline{\mathbf{p}_2\mathbf{p}_5}$ are lines of the Hilbert geometry $(\mathcal{H}, d_{\mathcal{H}})$ and of the hyperbolic geometry $(\mathcal{E}, d_{\mathcal{E}})$ too. Furthermore, the open segment $\overline{\mathbf{p}_3\mathbf{e}} = \mathcal{H} \cap \mathbf{p}_3\mathbf{p}_0$ is a line of the hyperbolic geometry $(\mathcal{E}, d_{\mathcal{E}})$, and the open segment $\overline{\mathbf{p}_3\mathbf{h}} = \mathcal{H} \cap \overline{\mathbf{p}_3\mathbf{p}_0}$ is a line of the Hilbert geometry $(\mathcal{H}, d_{\mathcal{H}})$. Moreover, $\overline{\mathbf{p}_3\mathbf{h}} \subset \overline{\mathbf{p}_3\mathbf{e}}$.

By the condition in the theorem, there is a Ceva triplet $(\mathbf{c}', \mathbf{a}', \mathbf{b}')$ of the trigon $\mathbf{abc}$ such that

(4) the triple $(\langle \mathbf{a}, \mathbf{b}, \mathbf{c}' \rangle_{\mathcal{H}}, \langle \mathbf{b}, \mathbf{c}, \mathbf{a}' \rangle_{\mathcal{H}}, \langle \mathbf{c}, \mathbf{a}, \mathbf{b}' \rangle_{\mathcal{H}})$ is of Ceva type.

Observe that $\langle \mathbf{a}, \mathbf{b}, \mathbf{c}' \rangle_{\mathcal{H}} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c}' \rangle_{\mathcal{E}}$ and $\langle \mathbf{c}, \mathbf{a}, \mathbf{b}' \rangle_{\mathcal{H}} = \langle \mathbf{c}, \mathbf{a}, \mathbf{b}' \rangle_{\mathcal{E}}$ by coincidence, and $\langle \mathbf{b}, \mathbf{c}, \mathbf{a}' \rangle_{\mathcal{H}} \neq \langle \mathbf{b}, \mathbf{c}, \mathbf{a}' \rangle_{\mathcal{E}}$ by Lemma 2.3. Then (4) implies that

(5) the triple $(\langle \mathbf{a}, \mathbf{b}, \mathbf{c}' \rangle_{\mathcal{E}}, \langle \mathbf{b}, \mathbf{c}, \mathbf{a}' \rangle_{\mathcal{E}}, \langle \mathbf{c}, \mathbf{a}, \mathbf{b}' \rangle_{\mathcal{E}})$ is *not* of Ceva type

what contradicts the hyperbolic Ceva's Theorem, and hence proves the theorem. □

Theorem 3.2 (Menelaus type characterization)**.** *If for every trigon* $\mathbf{abc}$ *in* $\mathcal{H}$ *there is a Menelaus triplet* $(\mathbf{c}', \mathbf{a}', \mathbf{b}')$ *such that the triple* $(\langle \mathbf{a}, \mathbf{b}, \mathbf{c}' \rangle, \langle \mathbf{b}, \mathbf{c}, \mathbf{a}' \rangle, \langle \mathbf{c}, \mathbf{a}, \mathbf{b}' \rangle)$ *is of Menelaus type, then the Hilbert geometry* $(\mathcal{H}, d_{\mathcal{H}})$ *is the hyperbolic geometry.*

This theorem can be proved in the same way as the previous one.

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